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HIGH-FREQUENCY LONG WAVE VIBRATIONS OF ELASTIC RODS

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The passage to approximate one-dimensional equations from the three-dimensional equations of elasticity is possible in the theory of rods exactly as in the theories of plates and shells if the displacements vary much more slowly in the longitudinal than in the transverse direction, i.e., in the case of long-wave vibrations in the longitudinal direction. In the low frequency case in dynamics, this condition results in the hypothesis of plane sections. Generally speaking, the hypothesis of plane sections is not true for high frequencies; the displacements can be rapidly oscillating functions of the transverse coordinates. The classical equations of the theory of rods, based on the hypothesis of plane sections, describe low-frequency vibrations. By using a variational asymptotic method below, displacement distribution over the section of a circular cylinder are found, and one-dimensional equations are derived in the case of free high-frequency long-wave vibrations of rods. These distributions have been obtained in [1] by asymptotic integration of the three-dimensional equations, however, the one-dimensional equations of high-frequency vibrations were not constructed there.

First-approximation equations in the geometrically nonlinear theory of anisotropic, inhomogeneous rods have been derived earlier [2], as have been the refined equations of rod bending vibrations [3] and the plate free high-frequency vibrations equations [4].

1. Let us consider the free vibrations of a rectilinear elastic rod of length $2l$ and constant cross section Ω. Let us select the axes of the Cartesian coordinate system x, x^α (the Greek superscripts take on the values 1, 2) so that the rod axis in the underformed state would coincide with the x-axis and the axes x^α would lie in the cross-sectional plane. The center of gravity is set at the origin. We denote the projections of the displacement vector on the x, x^α axes by w, w_α and the rod cross-sectional diameter by $2h$ $(h \ll l)$. We consider the rod ends to be rigidly fixed

$$w = w_\alpha = 0, \qquad x = \pm l. \tag{1.1}$$

The equations of free rod vibrations are extremals of the Lagrange functional [5]

$$I = \int_{t_1}^{t_2} dt \int_V \Lambda \, dV, \quad \Lambda = U - \frac{1}{2} \rho \left(w_{,t}^2 + w_{\alpha,t} w_{,t}^{\alpha}\right) \tag{1.2}$$

$$2U = \lambda (w_{,\alpha}{}^{\alpha})^2 + 2\lambda w_{,\alpha}{}^{\alpha} \, w_{,x} + (\lambda + 2\mu) \, w_{,x}{}^2 + 2\mu w_{(\alpha,\beta)} \, w^{(\alpha,\beta)} + \mu \, (w_{,\alpha} + w_{\alpha,x})(w^{,\alpha} + w_{,x}{}^{\alpha})$$

Here V is the volume occupied by the rod, ρ is the density, U the elastic energy, λ, μ the Lamé parameters. The comma in the subscripts denotes differentiation, and the parentheses denote the symmetrization operation.

2. We obtain the equations for the high-frequency vibrations by a variational asymptotic method [2].

Let us retain the asymptotically principal terms in h in the functional (1.2)

$$I = \Omega \int_{t_1}^{t_2} dt \int_{-l}^{l} \overline{\Lambda}\, dx \tag{2.1}$$

$$2\overline{\Lambda} = \langle \lambda (w_{,\alpha}{}^{\alpha})^2 + 2\mu w_{(\alpha,\beta)} w^{(\alpha,\beta)} + \mu w_{,\alpha} w^{,\alpha} - \rho (w_{,t}{}^2 + w_{\alpha,t} w_{,t}{}^{\alpha})\rangle$$

$$\langle A\rangle = \frac{1}{\Omega} \int_{\Omega} A\, d\Omega$$

The extremals of the functional (2.1) agree with the extremals of the functional

$$\int_{t_1}^{t_2} \overline{\Lambda} dt$$

Varying this latter, we obtain the equations

$$\lambda \frac{\partial}{\partial x^{\alpha}} w_{,\gamma}^{\gamma} + 2\mu \frac{\partial}{\partial x^{\beta}} w_{(\alpha,\beta)} - \rho w_{\alpha,tt} = 0 \tag{2.2}$$

$$\lambda w_{,\gamma}{}^{\gamma} \nu_{\alpha} + 2\mu w_{(\alpha,\beta)} \nu^{\beta} = 0 \quad \text{on } \Gamma$$

$$\mu \Delta w - \rho w_{,tt} = 0, \quad \partial w / \partial \nu = 0 \quad \text{on } \Gamma$$

Here Γ is the corss-sectional outline, ν^{α} are components of the external normal ν to the contour Γ, and Δ is the Laplace operator in the variable x^{α}.

Equations (2.2) govern two series of vibrations $F_{\perp}$ and $F_{\parallel}$ (the symbol $\perp$ is used if the transverse displacement w_{α} is very much greater than the longitudinal w, and the symbol $\parallel$ otherwise).

Let us assume that w, w_{α} depend on time according to the harmonic law

$$w = w^{\circ} e^{i\omega t}, \quad w_{\alpha} = w_{\alpha}{}^{\circ} e^{i\omega t}$$

We shall henceforth omit the superscript for w° and $w_{\alpha}{}^{\circ}$. The solutions of (2.2) have the form

$$F_{\perp}: \; w = 0, \quad w_{\alpha} = u f_{\alpha} (x^{\gamma}, \; \omega_{\perp}) \tag{2.3}$$

$$F_{\parallel}: \; w = \psi G (x^{\gamma}, \; \omega_{\parallel}), \quad w_{\alpha} = 0$$

It is convenient to write the expressions for f_{α} and G for a circular rod of radius h in the polar coordinates r, θ in the plane of the rod cross section

$$f_{\alpha} = h \left(f_r (r, \theta) \frac{x_{\alpha}}{r} + f_{\theta} (r, \theta) \frac{E_{\alpha\beta} x^{\beta}}{r}\right) \tag{2.4}$$

$$f_r (r, \theta) = \left[A J_n' (\alpha r) + \frac{n}{r} J_n (\beta r)\right] \cos n\theta$$

$$f_{\theta} (r, \theta) = -\left[A \frac{n}{r} J_n (\alpha r) + J_n' (\beta r)\right] \sin n\theta$$

$$A = \frac{2n\,(hJ_n'(\beta h) - J_n(\beta h))}{2hJ_n'(\alpha h) + (h^2\beta^2 - 2n^2)\,J_n(\alpha h)} \tag{2.5}$$

$$G = J_n(\beta r)\cos n\theta,\quad \alpha = \frac{\omega}{c_1},\quad \beta = \frac{\omega}{c_2},\quad c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_2 = \sqrt{\frac{\mu}{\rho}}$$

Here n is an integer, J_n is the Bessel function of the first kind of order n (the prime denotes the derivative with respect to r) and $E_{\alpha\beta}$ is the Levi-Civita tensor.

Therefore, $f_\alpha \sim 1,\ G \sim 1$.

The quantities u, ψ are functions of x, harmonic in time with the frequency ω, which are determined in each series from the appropriate dispersion equation

$$\begin{aligned} F_\perp:\quad & [2hJ_n'(\alpha h) + (\beta^2h^2 - 2n^2)\,J_n(\alpha h)][2hJ_n'(\beta h) + \\ & (\beta^2h^2 - 2n^2)\,J_n(\beta h)] = \\ & 4n^2\,[hJ_n'(\alpha h) - J_n(\alpha h)][hJ_n'(\beta h) - J_n(\beta h)] \\ F_\parallel:\quad & J_n'(\beta h) = 0 \end{aligned} \tag{2.6}$$

which are obtained from the condition of no stresses on the side surface of the rod.

In each series there is an infinite number of kinds of vibrations (branches) corresponding to different values of ω, the roots of the appropriate dispersion equation for each $n \geqslant 0$.

For $n = 0$ the dispersion equation (2.6) of the series $F_\perp$ reduces to the product of two factors. One yields the dispersion equation for the tension—compression vibrations over the thickness; hence the radial component of the displacement f_r is independent of θ and $f_\theta = 0$. The second factor yields the dispersion equation for torsional waves characterized by the presence of just one displacement component f_θ independent of θ.

For $n = 0$ the series $F_\parallel$ yields longitudinal axisymmetric vibrations. The corresponding dispersion equation has the form

$$J_1(\beta h) = 0$$

For $n \geqslant 1$ we have a family of bending waves which we shall examine later. In conformity with conditions (1.1), we consider that the functions are

$$u = \psi = 0,\quad x = \pm l \tag{2.7}$$

Without limiting the generality, we also consider that $u \sim \psi \sim 1$.

3. Let us first examine the series $F_\perp$. Considering u a given function of x we find w ($w = 0$ in a first approximation). Retaining only the principal terms containing w in (1.2) and the principal cross terms, we arrive at the functional

$$I_\perp = \frac{\Omega}{2}\int_{-l}^{l} \langle 2\lambda u f_{,\alpha}^{\ \alpha} w_{,x} + \mu\,(w_{,\alpha} + f_\alpha u_{,x})(w^{,\alpha} + f^\alpha u_{,x}) + \rho\omega^2 w^2 \rangle\, dx$$

whose variation will yield an equation and boundary condition for w

$$\begin{gathered} (\lambda + \mu)\,u_{,x} f_{,\alpha}^{\ \alpha} + \mu\Delta w + \rho\,\omega^2 w = 0 \\ \partial w / \partial \nu = -u_{,x} f^\alpha \nu_\alpha \quad \text{on}\quad \Gamma \end{gathered} \tag{3.1}$$

The solution of (3.1) has the form

$$w = u_{,x} g\,(x^\alpha)$$

For a circular rod (A is given by the first formula in (2.5))

$$g = h\left[AJ_n(\alpha r) - J_n(\beta r)\frac{nJ_n(\beta h) + 2AJ_n'(\alpha h)\,h}{hJ_n'(\beta h)}\right]\cos n\theta$$

Therefore, g is of othe order of h.

The next correction to w_α is found analogously: w is determined, and w_α is sought in the form $w_\alpha = uf_\alpha + w_\alpha'$, and the principal terms in w_α' in the principal cross terms are kept in the Lagrangean.

Because of the redefinition of u the constraint

$$\langle w_\alpha' f^\alpha\rangle = 0 \tag{3.2}$$

can be imposed on w_α'.

We arrive at equations governing the functions w_α'

$$\begin{aligned} &w_\alpha' = u_{,xx} g_\alpha \\ &\lambda\frac{\partial}{\partial x^\alpha} g_{,\gamma}^{\ \gamma} + 2\mu\frac{\partial}{\partial x^\beta} g_{(\alpha,\beta)} + \rho\omega^2 g_\alpha = \varkappa f_\alpha - (\lambda+\mu)\, g_{,\alpha} \\ &\lambda g_{,\gamma}^{\ \gamma}\nu_\alpha + 2\mu g_{(\alpha,\beta)}\nu^\beta = -\lambda g\nu_\alpha \quad \text{on} \quad \Gamma \end{aligned} \tag{3.3}$$

as a result of varying the functional under the condition (3.2).

Here $\varkappa$ is a Lagrange multiplier corresponding to the constraint (3.2). It is seen from (3.3) that $g_\alpha \sim h^2$. The explicit form of the solutions of these equations is not needed to construct the one-dimensional equations.

4. Formulas for the displacements of the second series are obtained by the same means. Considering ψ a given function of x, we find w_α ($w_\alpha = 0$ in a first approximation). Retaining only principal terms containing w_α and the principal cross terms in (1.2), we arrive at the functional

$$\begin{aligned} I_{\|} = \frac{\Omega}{2}\int_{-l}^{l} \langle &\lambda(w_{,\gamma}^{\ \gamma})^2 + 2\lambda w_{,\gamma}^{\ \gamma}\psi_{,x} G + 2\mu w_{(\alpha,\beta)} w^{(\alpha,\beta)} + \\ &2\mu\psi G_{,\alpha} w_{,x}^{\ \alpha} + \rho\omega^2 w_\alpha w^\alpha\rangle\, dx \end{aligned}$$

whose variation yields an equation and boundary conditions for w_α

$$\begin{aligned} &w_\alpha = \psi_{,x} G_\alpha\,(x^\beta) \\ &\lambda\frac{\partial}{\partial x^\alpha} G_{,\gamma}^{\ \gamma} + 2\mu\frac{\partial}{\partial x^\beta} G_{(\alpha,\beta)} + \rho\omega^2 G_\alpha = -(\lambda+\mu)\, G_{,\alpha} \\ &\lambda G_{,\gamma}^{\ \gamma}\nu_\alpha + 2\mu G_{(\alpha,\beta)}\nu^\beta = -\lambda G\nu_\alpha \quad \text{on} \quad \Gamma \end{aligned} \tag{4.1}$$

It follows from (4.1) that $G_\alpha \sim h$.

The solution for a circular rod has the form

$$\begin{aligned} &G_\alpha = \varphi_{,\alpha} + E_{\alpha\beta}\Psi_{,\beta} \\ &\varphi = \left[\frac{1}{\beta^2} J_n(\beta r) + MJ_n(\alpha r)\right]\cos n\theta \\ &\Psi = NJ_n\,(\beta r)\sin n\theta \end{aligned}$$

The constants M and N are determined from the boundary conditions on the side surface Γ

$$M = \frac{2J_n(\beta h)}{\beta^2} \frac{2n^2 - (n^2 - \beta^2 h^2)(2n^2 - \beta^2 h^2)}{J_n(\alpha h)\,[(2n^2 - \beta^2 h^2)^2 - 4n^2] + 2\beta^2 h^2 J_n'(\alpha h)} \tag{4.2}$$

$$N = \frac{2n}{\beta^2} \frac{\beta^2 h^2 J_n(\alpha h) + 2h(n^2 - \beta^2 h^2 - 1) J_n'(\alpha h)}{J_n(\alpha h)\,[(2n^2 - \beta^2 h^2)^2 - 4n^2] + 2\beta^2 h^3 J_n'(\alpha h)}$$

Furthermore, the correction to w is sought: it is assumed $w = \psi G + w'$, where ψ, G are considered known, and w' is found by variation of the functional in which the principal terms in w' and the principal cross terms are retained. Because of a redefinition of ψ the constraint

$$\langle G w' \rangle = 0 \tag{4.3}$$

can be imposed on w'.

We arrive at equations governing the function w'

$$w' = \psi_{,xx} H(x^\alpha) \tag{4.4}$$

$$\Delta H + \rho\omega^2 H = \varkappa^\circ G + \left(1 - \frac{\beta^2}{\alpha^2}\right) G_{,\alpha}^{\alpha}$$

$$\partial H / \partial \nu = -G_\alpha \nu^\alpha \text{ on } \Gamma$$

as a result of variating the functional under the condition (4.3).
Here $\varkappa^\circ$ is a Lagrange multiplier. It follows from (4.4) that $H \sim h^2$. The explicit form of the function H is not required for the construction of one-dimensional equations.

5. Now, let us assume that u, ψ are arbitrary functions of x and t. Evaluating $\langle \Lambda \rangle$ in (1.2) for each branch and keeping components of the order of h^{-2} and 1, we obtain

$$F_\perp:\ 2\langle \Lambda \rangle = \langle \lambda u^2 (f_{,\alpha}{}^\alpha)^2 + 2\lambda u u_{,xx} f_{,\alpha}{}^\alpha g + 2\mu u^2 f_{(\alpha,\beta)} f^{(\alpha,\beta)} + \mu u_{,x}{}^2 (f_\alpha + g_{,\alpha})(f^\alpha + g^{,\alpha}) - \rho(u_{,xt}^2 g^2 + u_{,t}{}^2 f_\alpha f^\alpha) \rangle \tag{5.1}$$

$$F_\parallel:\ 2\langle \Lambda \rangle = \langle \mu\psi^2 G_{,\alpha} G^{,\alpha} + \psi_{,x}{}^2 [\lambda(G + G_{,\alpha}{}^\alpha)^2 + 2\mu(G^2 + G_{(\alpha,\beta)} G^{(\alpha,\beta)} - G_{,\alpha} G^{,\alpha})] - \rho(\psi_{,xt}^2 G_\alpha G^\alpha + \psi_{,t}{}^2 G^2) \rangle$$

The average Lagrangean for the lowest branch of $F_\perp$ corresponding to $\omega_\perp = 0$ is not written down here since this branch is described by the classical Bernoulli – Euler equation.

Let us note that the functions g_α and H did not enter into the average Lagrangeans because of the constraints (3.2) and (4.3).

6. The vibrations corresponding to any two different branches with an error no greater than the magnitude of the terms kept in the Lagrangeans are orthogonal in the elastic and kinetic energies

$$\int_{-l}^{l} \langle \underset{1}{\sigma^{ij}} \underset{2}{\varepsilon_{ij}} \rangle\, dx = 0, \quad \int_{-l}^{l} \langle \underset{1}{w_{,t}{}^i} \underset{2}{w_{i,t}} \rangle\, dx = 0 \tag{6.1}$$

Here w_i, σ^{ij}, ε_{ij} are the displacements, stresses and strains, the indices 1 and 2 denote their values for different branches, and the indices i, j run through the values 0, 1, 2.

Equations (6.1) are verified by direct substitution. The equations and boundary conditions which the functions f_α, g, g_α and G, H, G_α satisfy, as well as the boundary conditions (2.7) at the rod ends are used here.

The orthogonality in the energy means that the vibrations corresponding to different branches are independent.

7. We obtain equations of the series $F_\perp$ and $F_\parallel$ of free high-frequency rod vibrations by varying the functionals (5.1). These equations have the form ($\omega_\perp$ and $\omega_\parallel$ are the roots of the appropriate dispersion equations (2.6)):

$$\rho a_\perp(\omega_\perp^2 u + u_{,tt}) = b_\perp u_{,xx} + \rho c_\perp u_{,xxtt} \tag{7.1}$$

$$\rho a_\parallel(\omega_\parallel^2 \psi + \psi_{,tt}) = b_\parallel \psi_{,xx} + \rho c_\parallel \psi_{,xxtt}$$

$$a_\perp = \Omega\langle f_\alpha f^\alpha\rangle,\quad b_\perp = \mu a_\perp + \rho\omega_\perp^2 c_\perp - (\lambda+\mu)d_\perp + \mu e_\perp$$

$$c_\perp = \Omega\langle g^2\rangle,\quad d_\perp = \Omega\langle f^{\alpha}_{,\alpha} g\rangle,\quad e_\perp = \int_\Gamma g f_\alpha \nu^\alpha d\Gamma$$

$$a_\parallel = \Omega\langle G^2\rangle,\ b_\parallel = (\lambda+2\mu)a_\parallel + \rho\omega_\parallel^2 c_\parallel + (\lambda+\mu)d_\parallel - \mu e_\parallel$$

$$c_\parallel = \Omega\langle G_\alpha G^\alpha\rangle,\quad d_\parallel = \Omega\langle G G^{\alpha}_{,\alpha}\rangle,\quad e_\parallel = \int_\Gamma G G_\alpha \nu^\alpha d\Gamma$$

For a circular rod of radius h

$$\pi^{-1}a_\perp = A^2\{hJ_n(\alpha h)J_n'(\alpha h) + {}^1/_2(\alpha^2h^2 - n^2)J_n^2(\alpha h) + {}^1/_2 h^2 J_n'^2(\alpha h)\} + 2AnJ_n(\alpha h)J_n(\beta h) + hJ_n(\beta h)J_n'(\beta h) + {}^1/_2(\beta^2h^2 - n^2)J_n^2(\beta h) + {}^1/_2 h^2 J_n'^2(\beta h)$$

$$\pi^{-1}c_\perp = A^2[(\alpha^2h^2 - n^2)J_n^2(\alpha h) + h^2J_n'^2(\alpha h)]/2\alpha^2 + B^2[(\beta^2h^2 - n^2)J_n^2(\beta h) + h^2J_n'^2(\beta h)]/2\beta^2 - 2ABh[J_n(\alpha h)J_n'(\beta h) - J_n(\beta h)J_n'(\alpha h)]/(\alpha^2 - \beta^2)$$

$$\pi^{-1}d_\perp = AB\alpha^2h[J_n(\alpha h)J_n'(\beta h) - J_n(\beta h)J_n'(\alpha h)]/(\alpha^2 - \beta^2) - {}^1/_2 A^2[(\alpha^2h^2 - n^2)J_n^2(\alpha h) + h^2J_n'^2(\alpha h)]$$

$$\pi^{-1}e_\perp = h[AJ_n(\alpha h) - BJ_n(\beta h)][AJ_n'(\alpha h) + n/h\, J_n(\beta h)]$$

$$B = \frac{n}{J_n'(\beta h)}\times \frac{4hJ_n'(\alpha h)J_n'(\beta h) - 2J_n(\beta h)J_n'(\alpha h) + h^{-1}(\beta^2h^2 - 2n^2)J_n(\alpha h)J_n(\beta h)}{2hJ_n'(\alpha h) + (\beta^2h^2 - 2n^2)J_n(\alpha h)}$$

$$\pi^{-1}a_\parallel = \frac{1}{2\beta^2}[(\beta^2h^2 - n^2)J_n^2(\beta h) + h^2J_n'^2(\beta h)]$$

$$\pi^{-1}d_\parallel = M\frac{\alpha^2h^2}{\alpha^2 - \beta^2}[J_n(\beta h)J_n'(\alpha h) - J_n(\alpha h)J_n'(\beta h)] - \frac{1}{2\beta^2}[(\beta^2h^2 - n^2)J_n^2(\beta h) + h^2J_n'^2(\beta h)]$$

$$\pi^{-1}e_\parallel = J_n(\beta h)\left[\frac{h}{\beta^2}J_n'(\beta h) + MhJ_n'(\alpha h) + NnJ_n(\beta h)\right]$$

$$\pi^{-1}c_\parallel = (N^2 + \beta^{-4})\{hJ_n'(\beta h)J_n(\beta h) + {}^1/_2[(\beta^2h^2 - h^2)J_n^2(\beta h) + h^2J_n'^2(\beta h)]\} + M^2\{hJ_n'(\alpha h)J_n(\alpha h) + {}^1/_2[(\alpha^2h^2 - n^2)J_n^2(\alpha h) +$$

$$h^2 J_n'^2(\alpha h)]\} + 2MNnh^{-1}J_n(\alpha h)J_n(\beta h) + 2Nnh^{-1}\beta^{-2}J_n^2(\beta h) +$$
$$\frac{2hM}{\beta^2(\alpha^2-\beta^2)}[\alpha^2 J_n(\alpha h)J_n'(\beta h) - \beta^2 J_n'(\alpha h)J_n(\beta h)]$$

The quantity A is defined by the first formula in (2.5), while M and N are defined by (4.2).

The approximation constructed in the paper has an error of order h. In conformity with this, the boundary conditions (1.1) are satisfied to the accuracy of terms of order h (since $w = u_{,x}g \sim h$). Therefore, the residual in the boundary conditions induces an error of the same order of smallness in the solution as does replacement of the equations of three-dimensional theory of elasticity by the approximate one-dimensional equations.

8. As an illustration of the application of (7.1), let us consider the problem of the high-frequency free vibrations of a rod of length $2l$, clamped rigidly at the ends. The solution of (7.1) under the boundary conditions

$$u(-l) = u(l) = \psi(-l) = \psi(l) = 0$$

is constructed by separation of variables. Consequently, we obtain

$$u_m = (A_m \cos\gamma_m t + B_m \sin\gamma_m t)\sin\lambda_m x \tag{8.1}$$
$$\gamma_m = \sqrt{\frac{\rho a\omega^2 + b\lambda_m^2}{\rho(a + c\lambda_m^2)}}, \quad \lambda_m = \frac{m\pi}{l}$$

where m is an integer, ω is a root of the dispersion equation (2.6), and a, b, c are coefficients in the first equation in (7.1).

The solution has the same form for the functions ψ_m but the coefficients a, b, c and the frequency ω are evaluated by formulas corresponding to the series $F_{\parallel}$.

The solution obtained has the following meaning. High frequency longwave harmonic vibrations with frequencies $\omega_{\perp}$ and $\omega_{\parallel}$ are possible in an infinite rod, the displacements are hence constant along the rod but vary as the functions $g(x^\alpha)$, $f_\alpha(x^\beta)$ or $G(x^\alpha)$, $G_\alpha(x^\beta)$ in the transverse coordinate. If the rod is clamped at the ends, then a slowly varying field along the axis is superposed on the constant displacement field along the longitudinal coordinate. The natural frequencies vary and will differ from ω. The second formula in (8.1) again yields the correction to ω. Hence, although the formal solution (8.1) is valid for any integer m, the values of m should be limited in such a way that γ_m evaluated by the second formula in (8.1) would not be too different from ω.

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